

On Efficient Implementation of Discontinuous Galerkin Method for Numerical Simulation of Two-Dimensional Gas Dynamic Flows on Unstructured Meshes

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Introduction

Gas dynamics specifics

- Discontinuity of solution
- Hydro and gas dynamic instabilities (Rayleigh — Taylor, Kelvin — Helmholtz, etc.)
- Direction of disturbances propagation in subsonic and supersonic flows

Methods

- **FDM** — only structured meshes & simple geometry
- **FEM** — unstructured meshes, continuous solution, high-order
- **FVM** — unstructured meshes, high numerical diffusivity, low-order

Discontinuous Galerkin method

Briefly: **FEM + FVM = DG**

Governing equations

Euler equations

$$\left. \begin{array}{l} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \\ \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div} \left(\rho \mathbf{v} \otimes \mathbf{v} + p \hat{\mathcal{I}} \right) = \mathbf{0}, \\ \frac{\partial e}{\partial t} + \operatorname{div} [(e + p) \mathbf{v}] = 0 \end{array} \right\} \quad \begin{array}{l} \frac{\partial \mathbf{U}}{\partial t} + \operatorname{div} \mathcal{F}(\mathbf{U}) = \mathbf{0}, \\ \mathcal{F}(\mathbf{U}) = \operatorname{diag}\{\mathbf{F}, \mathbf{G}, \mathbf{0}\} \end{array}$$

$$\mathbf{U} = [\rho, \rho u, \rho v, \rho w, e]^T,$$

$$\mathbf{F} = [\rho u, \rho u^2 + p, \rho uv, \rho uw, (e + p)u]^T,$$

$$\mathbf{G} = [\rho v, \rho vu, \rho v^2 + p, \rho vw, (e + p)v]^T.$$

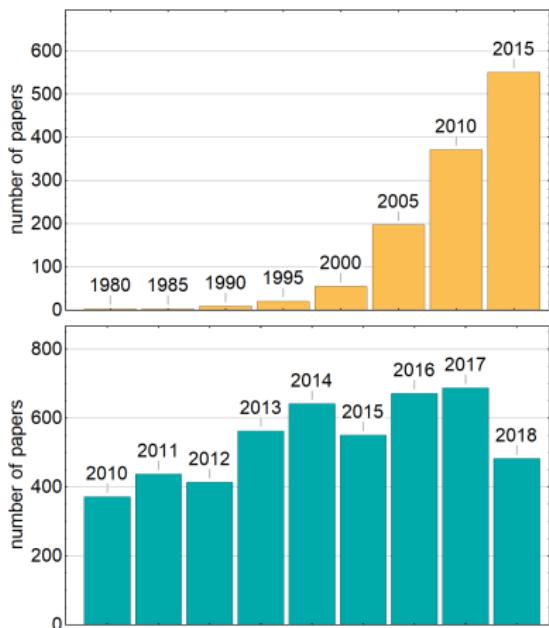
ρ — density, $\mathbf{v} = (u, v, w)^T$ — vector of velocity, p — pressure, $e = \rho \varepsilon + \rho \frac{\mathbf{v}^2}{2}$ — volumetric total energy.

EoS for perfect gas:

$$p = (\gamma - 1)\rho\varepsilon, \quad \gamma > 1.$$

Brief DG review

Publication statistic at Scopus



Usability

- No commercial DG codes
- First open-source implementations: 2017-18, e.g. HopeFOAM

Advantages

- Compact stencil
- Easy to increase the accuracy order

Main difficulties

- Monotonization of solution in case of strong discontinuities
- Implementation complexity

Runge–Kutta Discontinuous Galerkin method

Solution approximation on cell

$$\mathbf{U}_h(\vec{x}, t) = \sum_{j=1}^N \sum_{s=0}^m \mathbf{U}_j^{(s)}(t) \varphi_j^{(s)}(\vec{x})$$
$$\varphi_j^{(s)}(\vec{x}) \in \{ f(\vec{x}) : f|_{I_k} \in P^m(I_k), k = \overline{1, N} \}$$

Spatial discretization: Discontinuous Galerkin ODE system

$$\frac{d\mathbf{U}_j^{(r)}(t)}{dt} - \int_{I_j} \mathcal{F}_j \nabla \varphi_j^{(r)} dS + \oint_{\partial I_j} \mathcal{F}_j \cdot \mathbf{n} \varphi_j^{(r)} dl = 0$$

Time discretization: Runge–Kutta method

$$\mathbf{U}^* = \mathbf{U}^n + \tau \mathbf{L}_h(\mathbf{U}^n)$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{1}{2} \mathbf{U}^* + \frac{1}{2} \tau \mathbf{L}_h(\mathbf{U}^*)$$

Efficiency issue: monotonization approach

- Solutions may contain discontinuities.
- Discontinuities causes non-physical oscillations and possible blow-ups.
- Additional monotonization is required.

Limiters

- WENO_S
- Barth–Jespersen (BJ)

Characteristic decomposition

- ① Turn the solutions to the Riemann invariants: $\mathbf{W}_k = \Omega_L(\theta)\mathbf{U}_k$, $k \in S$;
- ② Apply the limiter to the characteristic variables;
- ③ Turn back to the conservative ones: $\tilde{\mathbf{U}}_j = \Omega_R(\theta)\tilde{\mathbf{W}}_j$.

Limiters: WENO for the compact stencil

- Original idea: combination of polynomials on stencil
- Simplification: modified polynomials on neighbour cells instead of interpolation polynomials
- Highly likely not decreasing an order of accuracy on local extrema

WENO Simple limiter (WENO_S)

	$I_{i,j+1}^2$	
$I_{i-1,j}^1$	$I_{i,j}^0$	$I_{i+1,j}^3$
	$I_{i,j-1}^4$	

$$\begin{aligned}\tilde{U}_0 &= \sum_k w_k p_k; \\ p_k &= U_k - U_k^{(0)} + U_0^{(0)}; \\ U_k^{(0)} &= \frac{1}{|I^0|} \int_{I^0} U_k(x, y) dx dy.\end{aligned}$$

Limiters: Barth–Jespersen (BJ)

- Evaluating the minimum and maximum cell means within the neighbouring cells:

$$m = \min_{k \in N} U_k^{(0)}, \quad M = \max_{k \in N} U_k^{(0)}.$$

- Estimating the coefficients at each limiting point:

$$y(\mathbf{x}_I) = \begin{cases} \frac{M - U^{(0)}}{U(\mathbf{x}_I) - U^{(0)}}, & U(\mathbf{x}_I) - U^{(0)} > 0, \\ \frac{m - U^{(0)}}{U(\mathbf{x}_I) - U^{(0)}}, & U(\mathbf{x}_I) - U^{(0)} < 0, \\ 1, & \text{otherwise.} \end{cases}$$

- Computing the correction coefficient:

$$\alpha = \min\{1, \min_I y(\mathbf{x}_I)\}.$$

- Correct the solution:

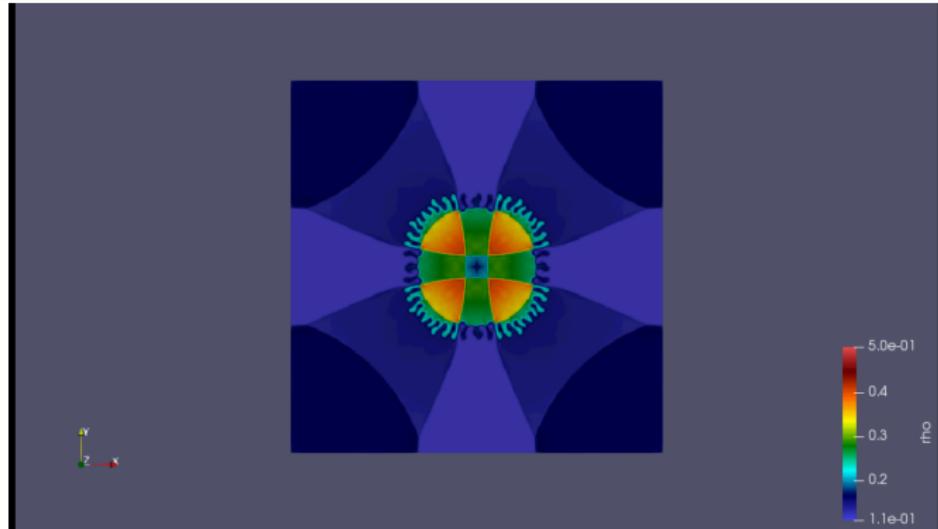
$$U = U^{(0)} + \alpha \nabla U \cdot (\vec{x} - \vec{x}_c).$$

Limiters comparison

Resolution check

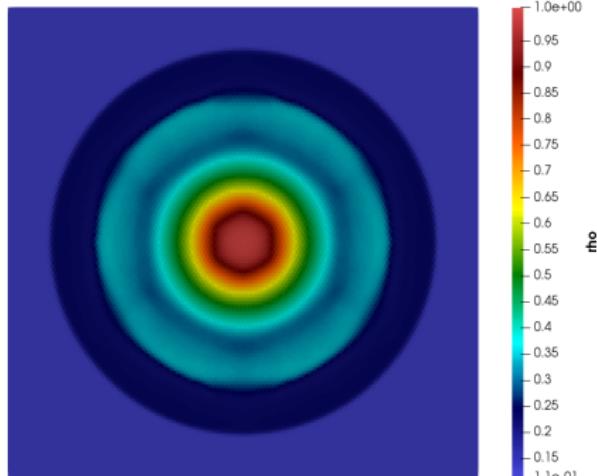
Cylindrical Sod problem

$$(\rho, u, v, w, p) = \begin{cases} (1, 0, 0, 0, 1), & r \leq 0.4; \\ (0.125, 0, 0, 0, 0.1), & r > 0.4. \end{cases}$$

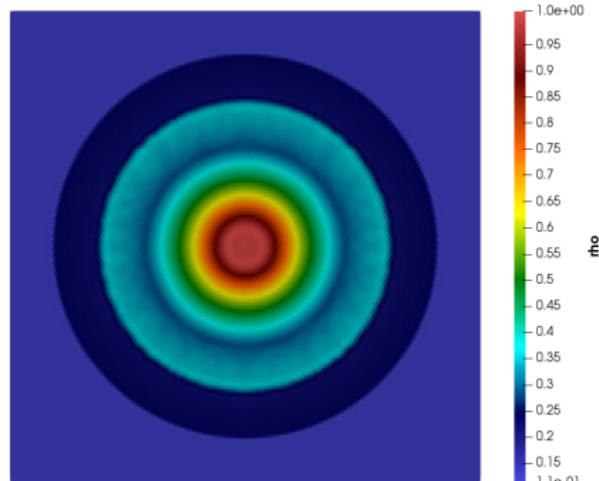


Limiters comparison

Resolution check



Barth–Jespersen limiter



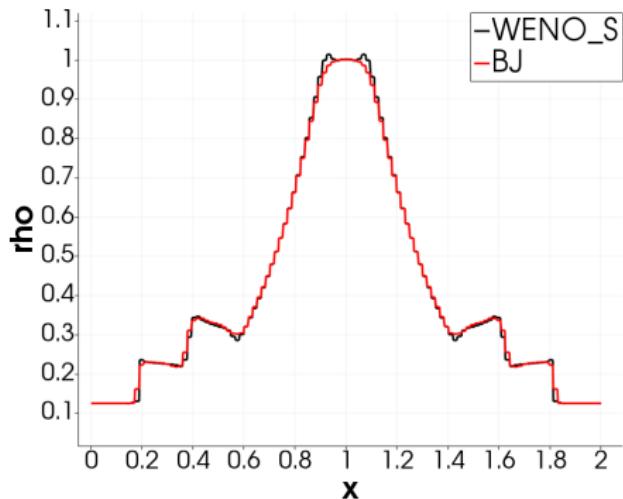
WENO_S limiter

Common settings

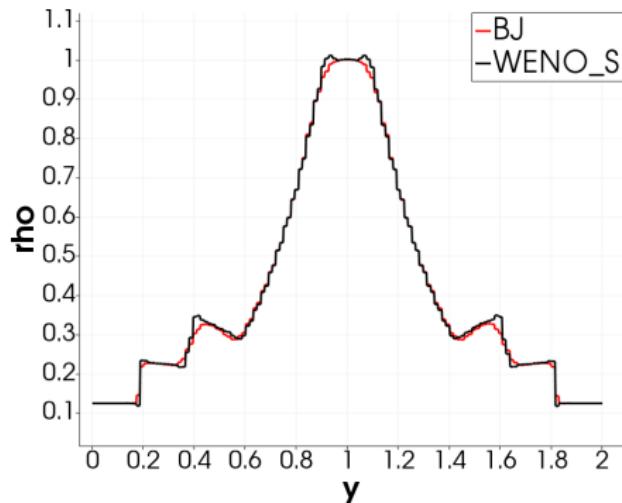
$t^* = 0.25$; ≈ 30000 triangles in 2×2 square; $\text{Co}_{\max} = 0.5$; Local Lax–Friedrichs numerical flux

Limiter comparison

Resolution check



Density field along $y = 0$ line



Density field along $x = 0$ line

Common settings

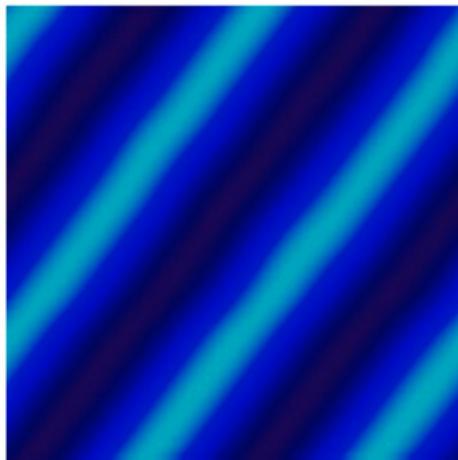
$t^* = 0.25$; ≈ 30000 triangles in 2×2 square; $\text{Co}_{\max} = 0.5$; Local Lax–Friedrichs numerical flux;

Limiter comparison

Accuracy analysis

Travelling wave

$$\begin{aligned}\rho &= \rho_0 - a \sin(wt - k(x - 0.75y)); \\ \rho_0 &= 1; \quad w = k = 2\pi; \quad a = 10^{-6}.\end{aligned}$$



Limiter comparison

Accuracy analysis

Approximation errors without any limiter and the BJ limiter

n	$\ \Delta_i\ _C \cdot 10^6$	$\frac{\ \Delta_i\ _C}{\ \Delta_{i-1}\ _C}$	$\ \Delta_i\ _{L_1} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_1}}{\ \Delta_{i-1}\ _{L_1}}$	$\ \Delta_i\ _{L_2} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_2}}{\ \Delta_{i-1}\ _{L_2}}$
12	0,32926	–	0,16212	–	0,11369	–
24	0,08589	3,83349	0,04112	3,94311	0,02912	3,90363
48	0,02172	3,95441	0,01031	3,98750	0,00736	3,97569
96	0,00563	3,85545	0,00259	3,96659	0,00184	3,97433

Approximation errors with the WENO_S limiter

n	$\ \Delta_i\ _C \cdot 10^6$	$\frac{\ \Delta_i\ _C}{\ \Delta_{i-1}\ _C}$	$\ \Delta_i\ _{L_1} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_1}}{\ \Delta_{i-1}\ _{L_1}}$	$\ \Delta_i\ _{L_2} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_2}}{\ \Delta_{i-1}\ _{L_2}}$
12	0,32988	–	0,16227	–	0,11365	–
24	0,08610	3,83121	0,04113	3,94528	0,02912	3,90286
48	0,02176	3,95696	0,01031	3,98839	0,00735	3,97541
96	0,00584	3,72300	0,00265	3,89019	0,00187	3,91765

Efficiency issue: time integration

Runge–Kutta method of 3rd order

$$\begin{aligned}\mathbf{U}^* &= \mathbf{U}^n + \tau \mathbf{L}_h(\mathbf{U}^n); \\ \mathbf{U}^{**} &= \frac{3}{4} \mathbf{U}^n + \frac{1}{4} \mathbf{U}^* + \frac{1}{4} \tau \mathbf{L}_h(\mathbf{U}^*); \\ \mathbf{U}^{n+1} &= \frac{1}{3} \mathbf{U}^n + \frac{2}{3} \mathbf{U}^{**} + \frac{2}{3} \tau \mathbf{L}_h(\mathbf{U}^{**}).\end{aligned}$$

Adams methods of 2nd and 3rd order

$$\begin{aligned}\mathbf{U}^{n+1} &= \mathbf{U}^n + \tau \left(\frac{3}{2} \mathbf{L}_h(\mathbf{U}^n) - \frac{1}{2} \mathbf{L}_h(\mathbf{U}^{n-1}) \right); \\ \mathbf{U}^{n+1} &= \mathbf{U}^n + \tau \left(\frac{23}{12} \mathbf{L}_h(\mathbf{U}^n) - \frac{4}{3} \mathbf{L}_h(\mathbf{U}^{n-1}) + \frac{5}{12} \mathbf{L}_h(\mathbf{U}^{n-2}) \right).\end{aligned}$$

Efficiency issue: time integration

Cylindrical Sod problem

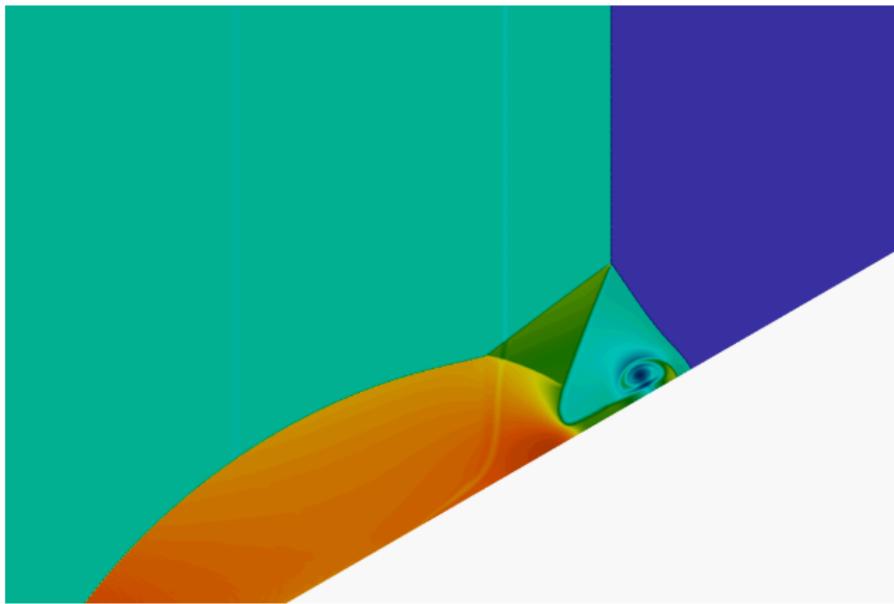
Common settings

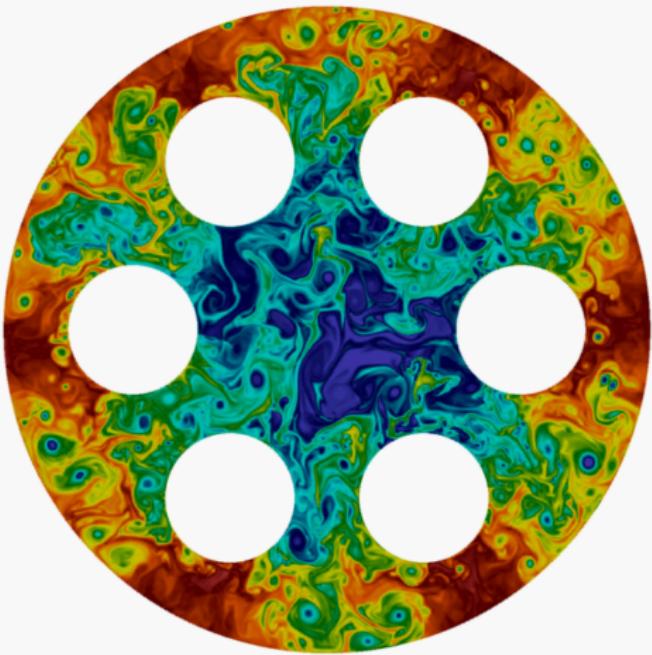
$t_{end} = 1.0$; ≈ 30000 triangles in 2×2 square; $Co_{max} = 0.5$; Local Lax–Friedrichs numerical flux; WENO_S limiter

	\bar{t}_τ, s	CFL_{max}	$T_{total}, CFL = 0.15$	$T_{total}, CFL = CFL_{max}$
RK-2	0.096	0.5	550	164
Adams-2	0.049	0.25	267	133
RK-3	0.142	0.5	724	229
Adams-3	0.052	0.15	273	273

Large mesh testing

Double Mach reflection





Thank you for your attention!