

# **On Efficient Implementation of Discontinuous Galerkin Method for Numerical Simulation of Two-Dimensional Gas Dynamic Flows on Unstructured Meshes**

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# Introduction

## Gas dynamics specifics

- Discontinuity of solution
- Hydro and gas dynamic instabilities (Rayleigh — Taylor, Kelvin — Helmholtz, *etc.*)
- Direction of disturbances propagation in subsonic and supersonic flows

## Methods

- **FDM** — only structured meshes & simple geometry
- **FEM** — unstructured meshes, continuous solution, high-order
- **FVM** — unstructured meshes, high numerical diffusivity, low-order

## Discontinuous Galerkin method

Briefly: **FEM** + **FVM** = **DG**

# Governing equations

## Euler equations

$$\left. \begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v} + p \hat{\mathcal{I}}) &= \mathbf{0}, \\ \frac{\partial e}{\partial t} + \operatorname{div}[(e + p) \mathbf{v}] &= 0 \end{aligned} \right\} \begin{aligned} \frac{\partial \mathbf{U}}{\partial t} + \operatorname{div} \mathcal{F}(\mathbf{U}) &= \mathbf{0}, \\ \mathcal{F}(\mathbf{U}) &= \operatorname{diag}\{\mathbf{F}, \mathbf{G}, 0\} \end{aligned}$$

$$\mathbf{U} = [\rho, \rho u, \rho v, \rho w, e]^T,$$

$$\mathbf{F} = [\rho u, \rho u^2 + p, \rho uv, \rho uw, (e + p)u]^T,$$

$$\mathbf{G} = [\rho v, \rho vu, \rho v^2 + p, \rho vw, (e + p)v]^T.$$

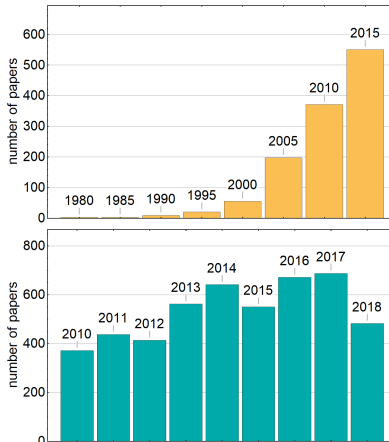
$\rho$  — density,  $\mathbf{v} = (u, v, w)^T$  — vector of velocity,  $p$  — pressure,  $e = \rho \varepsilon + \rho \frac{\mathbf{v}^2}{2}$  — volumetric total energy.

EoS for perfect gas:

$$p = (\gamma - 1)\rho \varepsilon, \quad \gamma > 1.$$

# Brief DG review

*Publication statistic at Scopus*



## Usability

- No commercial DG codes
- First open-source implementations: 2017-18, e.g. HopeFOAM

## Advantages

- Compact stencil
- Easy to increase the accuracy order

## Main difficulties

- Monotonization of solution in case of strong discontinuities
- Implementation complexity

# Runge–Kutta Discontinuous Galerkin method

## Solution approximation on cell

$$\mathbf{U}_h(\vec{x}, t) = \sum_{j=1}^N \sum_{s=0}^m \mathbf{U}_j^{(s)}(t) \varphi_j^{(s)}(\vec{x})$$

$$\varphi_j^{(s)}(\vec{x}) \in \{f(\vec{x}): f|_{I_k} \in P^m(I_k), k = \overline{1, N}\}$$

## Spatial discretization: Discontinuous Galerkin ODE system

$$\frac{d\mathbf{U}_j^{(r)}(t)}{dt} - \int_{I_j} \mathcal{F}_j \nabla \varphi_j^{(r)} dS + \oint_{\partial I_j} \mathcal{F}_j \cdot \mathbf{n} \varphi_j^{(r)} dl = 0$$

## Time discretization: Runge–Kutta method

$$\mathbf{U}^* = \mathbf{U}^n + \tau \mathbf{L}_h(\mathbf{U}^n)$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{1}{2} \mathbf{U}^* + \frac{1}{2} \tau \mathbf{L}_h(\mathbf{U}^*)$$

## Efficiency issue: monotonization approach

- Solutions may contain discontinuities.
- Discontinuities causes non-physical oscillations and possible blow-ups.
- Additional monotonization is required.

### Limiters

- WENO\_S
- Barth–Jespersen (BJ)

### Characteristic decomposition

- 1 Turn the solutions to the Riemann invariants:  $\mathbf{W}_k = \Omega_L(\theta)\mathbf{U}_k$ ,  $k \in S$ ;
- 2 Apply the limiter to the characteristic variables;
- 3 Turn back to the conservative ones:  $\tilde{\mathbf{U}}_j = \Omega_R(\theta)\tilde{\mathbf{W}}_j$ .

## Limiters: WENO for the compact stencil

- Original idea: combination of polynomials on stencil
- Simplification: modified polynomials on neighbour cells instead of interpolation polynomials
- Highly likely not decreasing an order of accuracy on local extrema

### WENO Simple limiter (WENO\_S)

	$I_{i,j+1}^2$	
$I_{i-1,j}^1$	$I_{i,j}^0$	$I_{i+1,j}^3$
	$I_{i,j-1}^4$	

$$\tilde{U}_0 = \sum_k w_k p_k;$$
$$p_k = U_k - U_k^{(0)} + U_0^{(0)};$$
$$U_k^{(0)} = \frac{1}{|I^0|} \int_{I^0} U_k(x, y) dx dy.$$

## Limiters: Barth–Jespersen (BJ)

- 1 Evaluating the minimum and maximum cell means within the neighbouring cells:

$$m = \min_{k \in N} U_k^{(0)}, \quad M = \max_{k \in N} U_k^{(0)}.$$

- 2 Estimating the coefficients at each limiting point:

$$y(\mathbf{x}_I) = \begin{cases} \frac{M - U^{(0)}}{U(\mathbf{x}_I) - U^{(0)}}, & U(\mathbf{x}_I) - U^{(0)} > 0, \\ \frac{m - U^{(0)}}{U(\mathbf{x}_I) - U^{(0)}}, & U(\mathbf{x}_I) - U^{(0)} < 0, \\ 1, & \text{otherwise.} \end{cases}$$

- 3 Computing the correction coefficient:

$$\alpha = \min\{1, \min_I y(\mathbf{x}_I)\}.$$

- 4 Correct the solution:

$$U = U^{(0)} + \alpha \nabla U \cdot (\vec{x} - \vec{x}_c).$$



# Limiters comparison

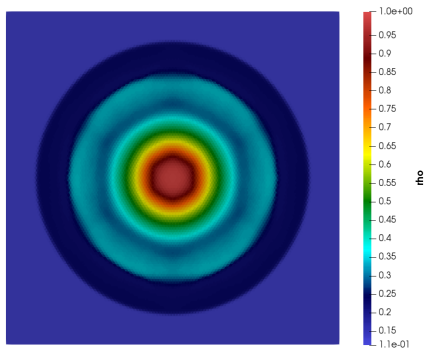
Resolution check

## Cylindrical Sod problem

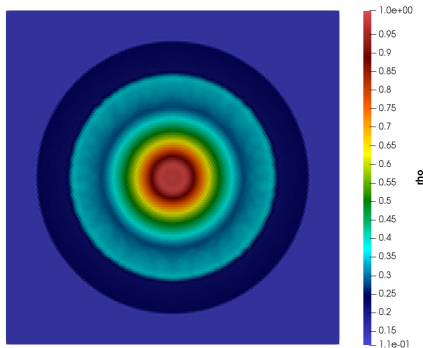
$$(\rho, u, v, w, p) = \begin{cases} (1, 0, 0, 0, 1), & r \leq 0.4; \\ (0.125, 0, 0, 0, 0.1), & r > 0.4. \end{cases}$$

# Limiters comparison

## Resolution check



*Barth-Jespersen limiter*



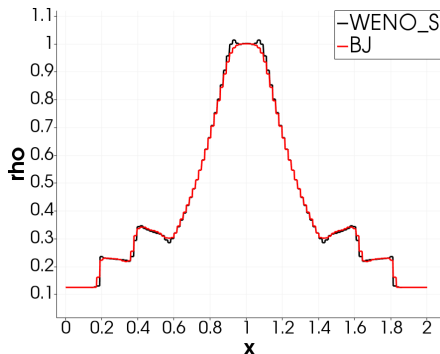
*WENO\_5 limiter*

## Common settings

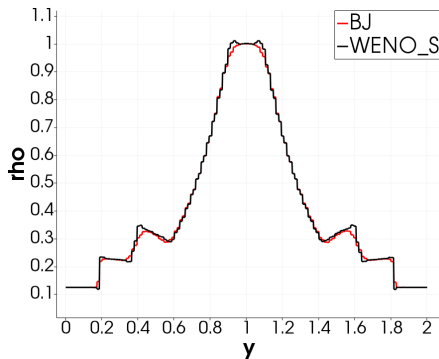
$t^* = 0.25$ ;  $\approx 30000$  triangles in  $2 \times 2$  square;  $Co_{max} = 0.5$ ; Local Lax-Friedrichs numerical flux

# Limiter comparison

## Resolution check



Density field along  $y = 0$  line



Density field along  $x = 0$  line

## Common settings

$t^* = 0.25$ ;  $\approx 30000$  triangles in  $2 \times 2$  square;  $Co_{max} = 0.5$ ; Local Lax–Friedrichs numerical flux;

# Limiter comparison

Accuracy analysis

## Travelling wave

$$\begin{aligned}\rho &= \rho_0 - a \sin(wt - k(x - 0.75y)); \\ \rho_0 &= 1; \quad w = k = 2\pi; \quad a = 10^{-6}.\end{aligned}$$

# Limiter comparison

## Accuracy analysis

Approximation errors without any limiter and the BJ limiter

$n$	$\ \Delta_i\ _C \cdot 10^6$	$\frac{\ \Delta_i\ _C}{\ \Delta_{i-1}\ _C}$	$\ \Delta_i\ _{L_1} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_1}}{\ \Delta_{i-1}\ _{L_1}}$	$\ \Delta_i\ _{L_2} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_2}}{\ \Delta_{i-1}\ _{L_2}}$
12	0,32926	—	0,16212	—	0,11369	—
24	0,08589	3,83349	0,04112	3,94311	0,02912	3,90363
48	0,02172	3,95441	0,01031	3,98750	0,00736	3,97569
96	0,00563	3,85545	0,00259	3,96659	0,00184	3,97433

Approximation errors with the WENO\_S limiter

$n$	$\ \Delta_i\ _C \cdot 10^6$	$\frac{\ \Delta_i\ _C}{\ \Delta_{i-1}\ _C}$	$\ \Delta_i\ _{L_1} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_1}}{\ \Delta_{i-1}\ _{L_1}}$	$\ \Delta_i\ _{L_2} \cdot 10^6$	$\frac{\ \Delta_i\ _{L_2}}{\ \Delta_{i-1}\ _{L_2}}$
12	0,32988	—	0,16227	—	0,11365	—
24	0,08610	3,83121	0,04113	3,94528	0,02912	3,90286
48	0,02176	3,95696	0,01031	3,98839	0,00735	3,97541
96	0,00584	3,72300	0,00265	3,89019	0,00187	3,91765

## Efficiency issue: time integration

### Runge–Kutta method of 3rd order

$$\begin{aligned} \mathbf{U}^* &= \mathbf{U}^n + \tau \mathbf{L}_h(\mathbf{U}^n); \\ \mathbf{U}^{**} &= \frac{3}{4} \mathbf{U}^n + \frac{1}{4} \mathbf{U}^* + \frac{1}{4} \tau \mathbf{L}_h(\mathbf{U}^*); \\ \mathbf{U}^{n+1} &= \frac{1}{3} \mathbf{U}^n + \frac{2}{3} \mathbf{U}^{**} + \frac{2}{3} \tau \mathbf{L}_h(\mathbf{U}^{**}). \end{aligned}$$

### Adams methods of 2nd and 3rd order

$$\begin{aligned} \mathbf{U}^{n+1} &= \mathbf{U}^n + \tau \left( \frac{3}{2} \mathbf{L}_h(\mathbf{U}^n) - \frac{1}{2} \mathbf{L}_h(\mathbf{U}^{n-1}) \right); \\ \mathbf{U}^{n+1} &= \mathbf{U}^n + \tau \left( \frac{23}{12} \mathbf{L}_h(\mathbf{U}^n) - \frac{4}{3} \mathbf{L}_h(\mathbf{U}^{n-1}) + \frac{5}{12} \mathbf{L}_h(\mathbf{U}^{n-2}) \right). \end{aligned}$$

# Efficiency issue: time integration

Cylindrical Sod problem

## Common settings

$t_{end} = 1.0$ ;  $\approx 30000$  triangles in  $2 \times 2$  square;  $Co_{max} = 0.5$ ; Local Lax–Friedrichs numerical flux; WENO\_S limiter

	$\overline{t_\tau}, s$	$CFL_{max}$	$T_{total}, CFL = 0.15$	$T_{total}, CFL = CFL_{max}$
RK-2	0,096	0.5	550	164
Adams-2	0,049	0.25	267	133
RK-3	0,142	0.5	724	229
Adams-3	0,052	0.15	273	273

# Large mesh testing

Double Mach reflection



**Thank you for your attention!**