On Efficient Implementation of Discontinuous Galerkin Method for Numerical Simulation of Two-Dimensional Gas Dynamic Flows on Unstructured Meshes

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Introduction

Gas dynamics specifics

- Discontinuity of solution
- Hydro and gas dymanic instabilities (Rayleigh — Taylor, Kelvin — Helmholtz, *etc.*)
- Direction of disturbances propagation in subsonic and supersonic flows

Methods

- **FDM** — only structured meshes & simple geometry
- **FEM** — unstructured meshes, continuous solution, high-order
- **FVM** — unstructured meshes, high numerical diffusivity, low-order

Discontinuous Galerkin method

Briefly: **FEM** + **FVM** = **DG**

Governing equations

Euler equations

$$\left.\begin{aligned}
&\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0,\\
&\frac{\partial\rho\mathbf{v}}{\partial t} + \mathrm{div}\left(\rho\mathbf{v}\otimes\mathbf{v} + p\hat{\mathcal{I}}\right) = \mathbf{0},\\
&\frac{\partial e}{\partial t} + \mathrm{div}\left[(e+p)\,\mathbf{v}\right] = 0
\end{aligned}\right\}
\begin{aligned}
&\frac{\partial\mathbf{U}}{\partial t} + \mathrm{div}\mathcal{F}(\mathbf{U}) = \mathbf{0},\\
&\mathcal{F}(\mathbf{U}) = \mathrm{diag}\{\mathbf{F},\mathbf{G},\mathbf{0}\}
\end{aligned}$$

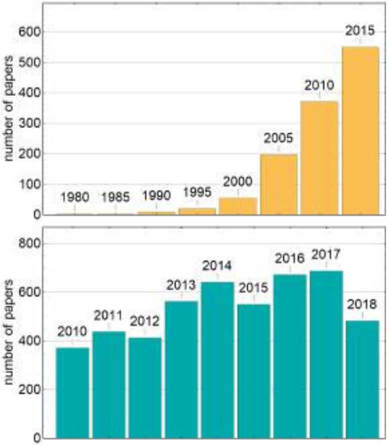
$$\begin{aligned}
\mathbf{U} &= [\rho,\ \rho u,\ \rho v,\ \rho w,\ e]^T,\\
\mathbf{F} &= [\rho u,\ \rho u^2+p,\ \rho uv,\ \rho uw,\ (e+p)u]^T,\\
\mathbf{G} &= [\rho v,\ \rho vu,\ \rho v^2+p,\ \rho vw,\ (e+p)v]^T.
\end{aligned}$$

ρ — density, $\mathbf{v} = (u,\ v,\ w)^T$ — vector of velocity, p — pressure, $e = \rho\varepsilon + \rho\frac{\mathbf{v}^2}{2}$ — volumetric total energy.

EoS for perfect gas: $$p = (\gamma-1)\rho\varepsilon, \quad \gamma > 1.$$

Brief DG review

Publication statistic at Scopus



Usability

- No commercial DG codes
- First open-source implementations: 2017-18, e.g. HopeFOAM

Advantages

- Compact stencil
- Easy to increase the accuracy order

Main difficulties

- Monotonization of solution in case of strong discontinuities
- Implementation complexity

Runge–Kutta Discontinuous Galerkin method

Solution approximation on cell

$$\mathbf{U}_h(\vec{x},\, t) = \sum_{j=1}^{N}\sum_{s=0}^{m}\mathbf{U}_j^{(s)}(t)\varphi_j^{(s)}(\vec{x})$$

$$\varphi_j^{(s)}(\vec{x}) \in \left\{\, f(\vec{x}) \colon\, f|_{I_k} \in P^m(I_k),\; k = \overline{1,\, N}\right\}$$

Spatial discretization: Discontinuous Galerkin ODE system

$$\frac{d\mathbf{U}_j^{(r)}(t)}{dt} - \int\limits_{I_j} \mathcal{F}_j \nabla\varphi_j^{(r)} dS + \oint\limits_{\partial I_j} \mathcal{F}_j \cdot \mathbf{n}\varphi_j^{(r)} dl = 0$$

Time discretization: Runge–Kutta method

$$\mathbf{U}^* = \mathbf{U}^n + \tau\mathbf{L}_h(\mathbf{U}^n)$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{1}{2}\mathbf{U}^* + \frac{1}{2}\tau\mathbf{L}_h\left(\mathbf{U}^*\right)$$

Efficiency issue: monotonization approach

- Solutions may contain discontinuities.
- Discontinuities causes non-physical oscillations and possible blow-ups.
- Additional monotonization is required.

Limiters

- WENO_S
- Barth–Jespersen (BJ)

Characteristic decomposition

1. Turn the solutions to the Riemann invariants: $\mathbf{W}_k = \Omega_L(\theta)\mathbf{U}_k,\ k \in S$;
2. Apply the limiter to the characteristic variables;
3. Turn back to the conservative ones: $\tilde{\mathbf{U}}_j = \Omega_R(\theta)\tilde{\mathbf{W}}_j$.

Limiters: WENO for the compact stencil

- Original idea: combination of polynomials on stencil
- Simplification: modified polynomials on neighbour cells instead of interpolation polynomials
- Highly likely not decreasing an order of accuracy on local extrema

WENO Simple limiter (WENO_S)

	$I^{2}_{i,j+1}$	
$I^{1}_{i-1,j}$	$I^{0}_{i,j}$	$I^{3}_{i+1,j}$
	$I^{4}_{i,j-1}$	

$$\tilde{U}_0 = \sum_k w_k p_k;$$

$$p_k = U_k - U_k^{(0)} + U_0^{(0)};$$

$$U_k^{(0)} = \frac{1}{|I^0|} \int_{I^0} U_k(x,y)\,dxdy.$$

Limiters: Barth–Jespersen (BJ)

1. Evaluating the minimum and maximum cell means within the neighbouring cells:

$$m = \min_{k \in N} U_k^{(0)}, \quad M = \max_{k \in N} U_k^{(0)}.$$

2. Estimating the coefficients at each limiting point:

$$y(\mathbf{x}_l) = \begin{cases} \dfrac{M - U^{(0)}}{U(\mathbf{x}_l) - U^{(0)}}, & U(\mathbf{x}_l) - U^{(0)} > 0, \\ \dfrac{m - U^{(0)}}{U(\mathbf{x}_l) - U^{(0)}}, & U(\mathbf{x}_l) - U^{(0)} < 0, \\ 1, & \text{otherwise.} \end{cases}$$

3. Computing the correction coefficient:

$$\alpha = \min\{1, \min_l y(\mathbf{x}_l)\}.$$

4. Correct the solution:

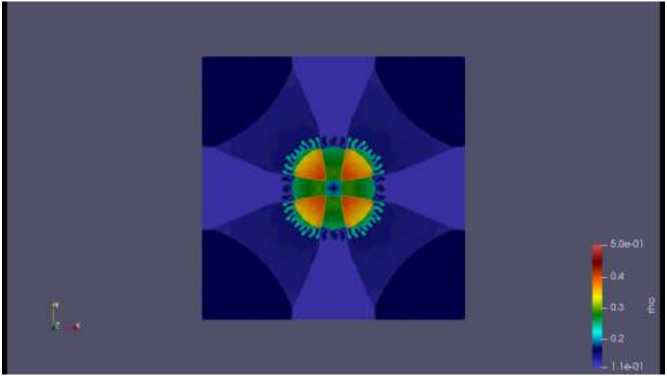
$$U = U^{(0)} + \alpha \nabla U \cdot (\vec{x} - \vec{x}_c).$$

Limiters comparison

Resolution check

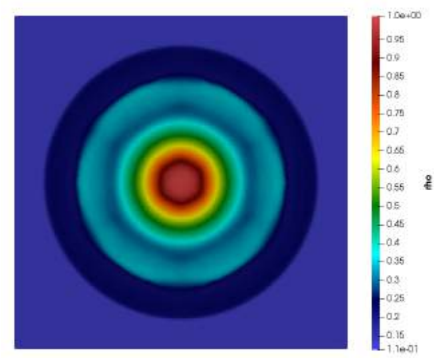
Cylindrical Sod problem

$$(\rho, u, v, w, p) = \begin{cases} (1, 0, 0, 0, 1), & r <= 0.4; \\ (0.125, 0, 0, 0, 0.1), & r > 0.4. \end{cases}$$

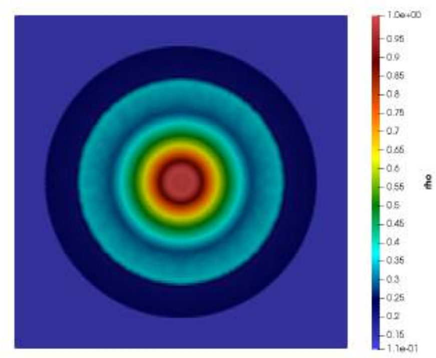


Limiters comparison

Resolution check



Barth–Jespersen limiter



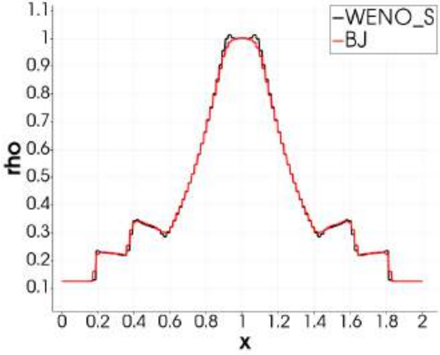
WENO_S limiter

Common settings

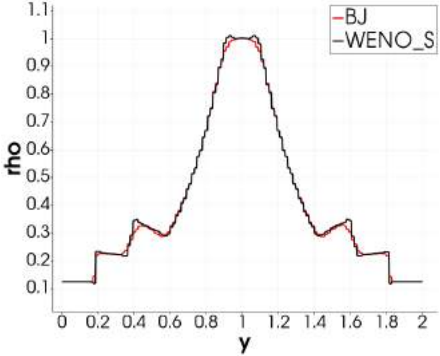
$t^* = 0.25$; $\approx$30000 triangles in 2×2 square; $\mathrm{Co}_{max} = 0.5$; Local Lax–Friedrichs numerical flux

Limiter comparison

Resolution check



Density field along $y = 0$ line



Density field along $x = 0$ line

Common settings

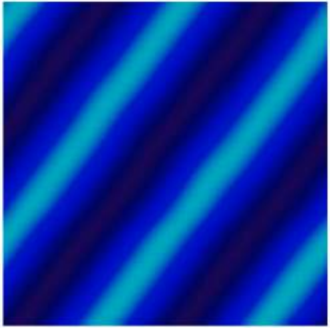
$t^* = 0.25$; $\approx$30000 triangles in 2×2 square; $\mathrm{Co}_{max} = 0.5$; Local Lax–Friedrichs numerical flux;

Limiter comparison

Accuracy analysis

Travelling wave

$$\rho = \rho_0 - a\sin(wt - k(x - 0.75y));$$
$$\rho_0 = 1; \quad w = k = 2\pi; \quad a = 10^{-6}.$$



Limiter comparison

Accuracy analysis

Approximation errors without any limiter and the BJ limiter

n	$\|\Delta_i\|_C \cdot 10^6$	$\frac{\|\Delta_i\|_C}{\|\Delta_{i-1}\|_C}$	$\|\Delta_i\|_{L_1} \cdot 10^6$	$\frac{\|\Delta_i\|_{L_1}}{\|\Delta_{i-1}\|_{L_1}}$	$\|\Delta_i\|_{L_2} \cdot 10^6$	$\frac{\|\Delta_i\|_{L_2}}{\|\Delta_{i-1}\|_{L_2}}$
12	0,32926	–	0,16212	–	0,11369	–
24	0,08589	3,83349	0,04112	3,94311	0,02912	3,90363
48	0,02172	3,95441	0,01031	3,98750	0,00736	3,97569
96	0,00563	3,85545	0,00259	3,96659	0,00184	3,97433

Approximation errors with the WENO_S limiter

n	$\|\Delta_i\|_C \cdot 10^6$	$\frac{\|\Delta_i\|_C}{\|\Delta_{i-1}\|_C}$	$\|\Delta_i\|_{L_1} \cdot 10^6$	$\frac{\|\Delta_i\|_{L_1}}{\|\Delta_{i-1}\|_{L_1}}$	$\|\Delta_i\|_{L_2} \cdot 10^6$	$\frac{\|\Delta_i\|_{L_2}}{\|\Delta_{i-1}\|_{L_2}}$
12	0,32988	–	0,16227	–	0,11365	–
24	0,08610	3,83121	0,04113	3,94528	0,02912	3,90286
48	0,02176	3,95696	0,01031	3,98839	0,00735	3,97541
96	0,00584	3,72300	0,00265	3,89019	0,00187	3,91765

Efficiency issue: time integration

Runge–Kutta method of 3nd order

$$\boldsymbol{U}^* = \boldsymbol{U}^n + \tau \boldsymbol{L}_h(\boldsymbol{U}^n);$$
$$\boldsymbol{U}^{**} = \frac{3}{4}\boldsymbol{U}^n + \frac{1}{4}\boldsymbol{U}^* + \frac{1}{4}\tau \boldsymbol{L}_h(\boldsymbol{U}^*);$$
$$\boldsymbol{U}^{n+1} = \frac{1}{3}\boldsymbol{U}^n + \frac{2}{3}\boldsymbol{U}^{**} + \frac{2}{3}\tau \boldsymbol{L}_h(\boldsymbol{U}^{**}).$$

Adams methods of 2nd and 3rd order

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \tau\left(\frac{3}{2}\mathbf{L}_h(\mathbf{U}^n) - \frac{1}{2}\mathbf{L}_h\left(\mathbf{U}^{n-1}\right)\right);$$
$$\mathbf{U}^{n+1} = \mathbf{U}^n + \tau\left(\frac{23}{12}\mathbf{L}_h(\mathbf{U}^n) - \frac{4}{3}\mathbf{L}_h\left(\mathbf{U}^{n-1}\right) + \frac{5}{12}\mathbf{L}_h\left(\mathbf{U}^{n-2}\right)\right).$$

Efficiency issue: time integration

Cylindrical Sod problem

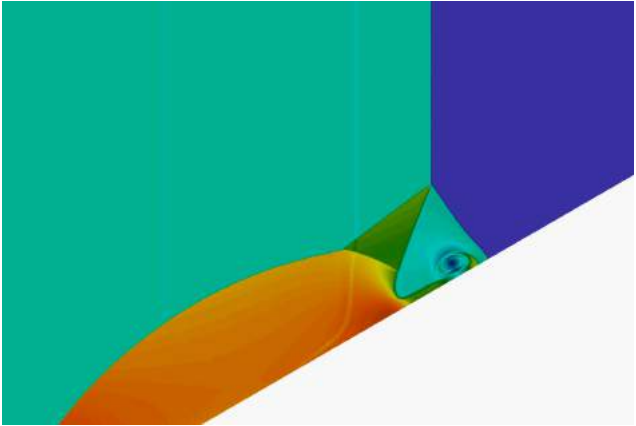
Common settings

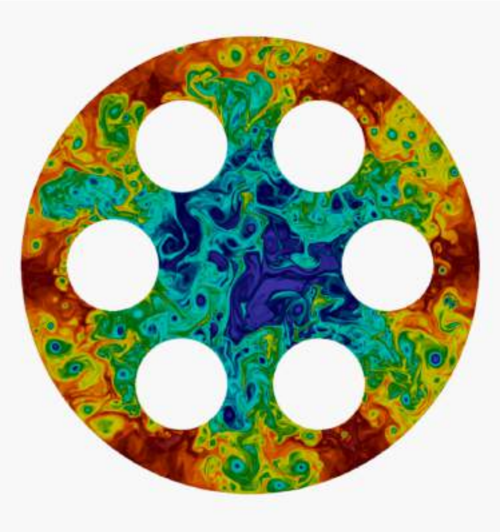
$t_{end} = 1.0$; $\approx$30000 triangles in 2×2 square; $Co_{max} = 0.5$; Local Lax–Friedrichs numerical flux; WENO_S limiter

	$\overline{t_\tau}$, s	$\mathrm{CFL_{max}}$	T_{total}, CFL $= 0.15$	T_{total}, CFL $= \mathrm{CFL_{max}}$
RK-2	0,096	0.5	550	164
Adams-2	0,049	0.25	267	133
RK-3	0,142	0.5	724	229
Adams-3	0,052	0.15	273	273

Large mesh testing

Double Mach reflection





Thank you for your attention!